

PRIVATE EXPERIENCE IN ADAPTIVE LEARNING MODELS*

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ABSTRACT

Here I provide a model that gives some insights regarding questions about actual economic behavior. I take as a source for stylized facts the experiments conducted by Marimon and Sunder as reported in *Econometrica*, 1993, in which it is shown that people initially do not behave according to the rational expectations assumption, but eventually learn to do so. I propose a slight generalization of the adaptive learning model in order to explain, besides the long run equilibrium observed, the stochastic-like time paths in the aggregate variables. In fact, the introduction of heterogeneity in private experience accumulated over time in a simple adaptive model with fixed decision rules is shown to be necessary and sufficient to generate the complex kind of dynamics present in the experiments. In our version of the Marcet–Sargent OLS model, people can not be using useful public information available, but only private experience instead, when they do price forecasting. Otherwise, we would not be able to explain the data with this model. This result sheds light on the experimental results, in the sense of suggesting a stronger degree of bounded rationality in experimental subjects. In addition, I provide examples within the proposed environment that improve upon the explanatory power of existing adaptive learning models.

Key words: Learning, Experience, Participation.

1 Introduction

Learning models in game theory and macroeconomics have been developed as an alternative to models in which the decision makers are assumed to have, in some specific sense, too much knowledge and rationality. Often, games and economic environments have complex settings. The actions of the agents involved, in the defined equilibria, require considerable amounts of information about those worlds in which agents act and interact. In addition, the solution concepts, or equilibria, usually require high levels of computational ability by the decision makers, who are also assumed to be optimizers. Besides this, additional problems associated with traditional game theory and general equilibrium models involve the issues of multiplicity of equilibria and equilibrium selection. Bayesian learning models, such as the one by Kalai and Lehrer ([8],[9],[10]) can be used to address the problems of equilibrium selection and learning in models with fully rational agents. But the same problems, together with the problems of excessive rationality, can be handled simultaneously by some learning models with boundedly rational agents, like the ones surveyed by Sargent ([18]). The last kind of models are more successful regarding experimental data.

The bulk of these analytical learning models have been developed with a focus mainly on convergence issues. Little concern has been given, for example, to the economic meaning of the specific limits or psychological insights regarding agents actual decisions. In the meanwhile, a body of evidence has been gathered both on the lack of rationality by experimental subjects in specific, static situations (for example in [11]), and also on learning in dynamic settings (as in [2], [15] and [19], for example) both in games and general equilibrium settings. This evidence in actual learning justifies the study of convergence issues; nevertheless, little has been done to match with analytical learning models stylized facts on transitional learning periods, as well as on the specific limits of the converging processes. The scarce existing literature on that matching in general equilibrium models focuses, as far as I know, on the use of learning models with representative agents.

One attempt at matching the learning evidence has been made by Marimon and Sunder (1993) ([15]). They conducted experiments in an environment *without exogenous uncertainty* in order to observe how much the data reflected the rational expectations hypothesis, and to address the problem of multiplicity of equilibria and equilibrium selection in that environment. They followed the research program established by Lucas ([12]), where it is conjectured that learning the rational expectations equilibrium where the *classical* monetary policy recipe (to be explained below) works will occur over time. Although they successfully address those problems, the kinds of models put forward by them (learning models with agents using all public information: the Marcet–Sargent adaptive model) do not capture all the features observed in the experimental data, such as the erratic oscillations in aggregate variables. In fact, their main interest is in reporting the observation that agents are not rational, but seem to learn to behave rationally and to choose the classical rational expectations equilibrium as time goes on. The general objective of this paper is to develop an appropriate model that allows us to better understand the complexity of the decision process that generates the learning and erratic oscillations observed in the experiments and, at the same time, model economic agents behavior that might be present in more general contexts.

The Genetic Algorithm (*GA*) has been used by Arifovic ([3]) with a similar purpose. Arifovic's

adaptation of the *GA* indeed reproduces the erratic oscillations and equilibrium selection that we want to match, but it does so by assuming that the individuals in the experiment are like genes that require crucially a kind of interaction with other individuals within a population (*crossover, election, mutation and reproduction*, the typical operations of the *GA*) that is not present in the experimental design under consideration. A proper use of the *GA* to explain the dynamics of the experiment at hand would require the use of a *GA* within the mind of each individual, and this would require extraordinary amounts of memory and computational ability, qualities that are typically lacking in experimental subjects, as demonstrated in other contexts.

I present my model as an Adaptive Algorithm (*AA*) alternative that requires remarkable simplicity in the decision process that takes place in the minds of the individuals and still generates the limits and kind of complex dynamics observed in the experimental data, without violating the experimental design and without using the features of experimenting, error or chaotic models¹. In fact, when those individuals use a simple constant decision rule that takes into account past personal experience, we get a model that sufficiently delivers the features that a proper hypothetical adaptation of the *GA* would produce. It is this intuitive way of modeling behavior that points to an interesting new interpretation of the randomness present in the experimental data: further bounded rationality.

The key feature that I introduce allows agents to acquire different amounts of experience: They might not participate in the economy in every period. The intuitive idea of this *participation* model is based in a real-life (as well as experimental design) feature of goods markets which operate every day: Some people go there every day, some every other day, some weekly, erratically, etc. But even if people do not go every day to market, though, they can observe everyday market-clearing prices. Now, although the size of the market remains roughly constant over time, each participating person in that market uses idiosyncratic habits. So, even though the size of the market is constant, its composition is not homogeneous in many possible ways. The diversity of experiences used by the participating agents alone could potentially generate, even if they use time invariant decision rules, complex dynamics. This might happen, for instance, if people *cared more*, in some sense, about the information they get when they actually go to the market, than for the information they are able to glean from the newspapers, say, even if that is not an entirely rational behavior from the modeler's point of view.

I in fact show below that, after guaranteeing certain uniformity at the economy-wide level in this framework, and under the assumption of simple adaptive rules, the only possibility for the outcome of deterministic, smooth time paths in the aggregate variables in this type of economy is that every participating agent uses only public information when he or she is making decisions. As a corollary, I am able to infer that only when agents fail to use all the public information freely available to them, their decision rules produce stochastic time paths in aggregate variables. It follows that the reason representative agent adaptive learning models do not appropriately capture the randomness observed in the experimental data is because they implicitly assume that agents

¹ Other forms of heterogeneity, like the use of different individual decision rules over time by the different agents due to experimenting, unintended errors, etc., will also produce the desired effects. What is most probably taking place is the mixture of experimentation, on the one hand, and the use of private experience, on the other. This paper isolates the last aspect of reality, which has an intuitive appeal, and has not been taken into consideration before. Fixed rules that produce chaotic behavior are ruled out by the fact that the paths are not convergent.

use all information available to them (or participate every day, which is very unlikely). So, if agents use the proposed model, the experimental results show that they are not only not rational in the sense of both ignoring the future and lacking deep computational ability. In our version of the Marcet–Sargent model for example, people do not use revealing public information available for free, but past private experience instead, regarded (erroneously) as more relevant for them, when they make decisions that affect their future. This result sheds light for a possible additional interpretation of the experimental data, in the sense of pointing to a further level of bounded rationality in experimental subjects.

In order to complete the picture, I illustrate using simulations, that even with heterogeneous participation, our version of the Marcet–Sargent model does not seem to be the most useful one for calibration of the erratic paths observed in the data. The reason is because the underlying deterministic process is shown not to be rich enough to produce the desired level of oscillations: it is monotonically convergent after a few initial periods. This problem is solved by introducing decision rules whose underlying deterministic paths might be monotonic or oscillatory, depending on the learning rate chosen. An example within the class of adaptive models that uses the feature of private experience use and also exhibits oscillatory paths when people are homogeneous, is the proposed *Direct Savings Estimation (DSE)* model. In this model agents are not maximizers, but approximate maximization, instead, in a myopic way. The rule agents use is simply to increase (decrease) savings when the previous savings decision was too low (high) ex-post. Comparisons between the simulations of the Marcet–Sargent model with heterogeneous agents, and the *DSE* model, illustrate graphically the better potential of the former model for calibration purposes. At the end of the paper, we point out the need of a more careful study, taking into account the actual data and more experimentation².

2 Experimental Evidence

Before going into the model, and for the sake of completeness, I will describe the design and results of the experiments in Marimon-Sunder [15]. The authors design an experiment trying to replicate the environment of the standard fiat money Overlapping Generations (OLG) theoretical model. They select a group of people that remain throughout the duration of a given experimental session. In each period of the experiments, a subgroup is selected randomly from that group of people to participate in the experimental activities for two consecutive periods, after which they return to the original group to await a new chance to participate. Each individual in the population remains for the whole duration of the experiment, in front of a computer terminal, receiving information and entering own decisions. Direct communication among experimental subjects is not allowed. Once a subgroup is selected in period t , each member is provided with an endowment w_1 of tokens,

² An unavoidable requirement for calibration or estimation is the use of data on individual behavior, unfortunately not available from the experiments referred to.

which will play the role of consumption goods. During their second period of activity, the members of the subgroup will be provided with an endowment w_2 of tokens, and, once their turn is over, they will be rewarded, in dollars, by the quantity $k(\ln(c_t^1) + \beta \ln(c_t^2))$, where k is a given constant, and c_t^1 and c_t^2 are the numbers of tokens individuals retain, after transactions, at the first and second periods of their activity, respectively. Although tokens cannot be transferred from period to period (they are *perishable goods*), *francs*, the money in that economy, are transferable. The market clearing price is set by a mechanism that asks *young* participants to submit hypothetical token supplies at a number of prices. After these hypothetical supplies are made continuous by interpolation (and subjects agree to supply at the corresponding intermediate points when the final price is set) and aggregated, the equilibrium price results from intersecting that supply with the demand of old consumers and the government. Every agent is informed of the prices that have cleared the markets, beginning at the initial period, regardless of turns of participation.

The experimental results show that inflation seems to converge or to stay around the low inflation stationary state (*LSS*) equilibrium. In particular, no non-stationary rational expectations equilibrium path is observed. There is no convergence to the high inflation stationary state (*HSS*) either, contrary to the rational expectations OLG model prediction that this state is stable. The findings are basically consistent for the 13 experimental economies. Together with the graphs of the actual paths, they present the (non stationary) path that would occur in the OLG perfect foresight model if the economy started at the initial actual inflation, and also the path that the economy would follow if the agents used recursive least squares as an adaptive rule following the actual initial inflation (the *Marcel- Sargent* path). These graphs show that people might be using some adaptive behavioral rule, and that the recursive ordinary least squares model reflects more the long run tendencies of the actual paths, contrary to the perfect foresight model, even though it does not reflect the erratic ups and downs of those paths.

3 The Model

3.1 The Basic OLG Model

In presenting the benchmark model, an OLG model with fiat money and without exogenous uncertainty, our purpose is to make a self-contained presentation without going into all the details of standard results. In each period, n agents come to participate in the economy during two consecutive periods. When they enter, they interact with the n people that entered in the previous period. It is common knowledge that people will receive w_1 of the only good in the economy in their first period of participation, and w_2 of it in their second period. There is no production or storage technology in the economy; in addition, the absence of credit markets makes fiat money the only asset in the economy capable of distributing goods inter temporally for any given individual.

In fact, the parameters of the economy will be set up in such a way as to make highly desirable for an individual to save part of his endowment in the first period of participation. The only way to do this in this economy will be to sell in exchange for money part of that endowment, at a market fixed price. The agents who will want to buy those goods are the current old people, who have money in their hands due to their own past savings decisions, and also the government, which is assumed in the present model to issue the amount of money necessary to finance, at the current market price, a given real deficit in that period. In the kind of equilibria to be defined, fiat money will generally have value (the prices of goods in terms of money will be finite), and young agents will be able to save, and then consume satisfactorily in their old age.

To complete the description of the kind of environment I want to start with, I will fix w_1 and w_2 such that $w_2 \ll w_1$. In this paper the (common knowledge) *ex-post* preferences for individual i of generation t are represented by the utility function $\ln(c_t^{i1}) + \beta^i \ln(c_{t+1}^{i2})$, for i , where $\beta^i \in \mathbf{R}_+$ is a discount factor, and c_t^{i1} , c_{t+1}^{i2} represent consumption in the first and second periods of participation, respectively. As it was said in Section 2, this is the reward that in fact individuals will receive at the end of the second period, depending on what c_t^{i1} and c_{t+1}^{i2} they decide and are able to hold. It is assumed that the government has to balance its budget by financing each period a real deficit per young agent in the economy of d through seignorage, emitting fiat money to do that. In the first period, when $t = 0$, an amount h_{-1} of units of money is given to each of the n initial old people.

To define the weakest form of equilibrium, I will require only that agents satisfy their budget constraints and that markets clear. I next present what is required in this model for this conditions to hold. Let $(p_t)_{t=0}^\infty$ be a given price process for the described economy in some equilibrium. That is, p_t represents the amount of fiat money at which one unit of the good in the economy can be traded in the market at period t , $t = 1, 2, \dots$. Then, the government's balanced budget of period t implies that the increment in per-young agent money supply in the economy, $h_t - h_{t-1}$, equals the per-young agent nominal deficit in that period at those prices, $p_t d$, or: $h_t - h_{t-1} = p_t d$, equation which, for h_{-1} given, and for all t defines $(h_t)_{t=1}^\infty$. In fact, the money that will be offered to each young agent of generation t for the part of w_1 that he will save amounts to h_{t-1} , which represents what each old agent has as his savings, plus $p_t d$, the new money per young agent that the government issues to finance its per-young agent deficit. Define the per-young agent's real money supply at t as m_t ; then, from the previous equation, I have that, for $t \geq 1$, $m_t = \frac{m_{t-1}}{\pi_t} + d$, where π_t , the rate of inflation at t is defined by $\pi_t \equiv p_t / p_{t-1}$.

Let s_t^i be the amount of real savings decided by young agent i of generation t ($s_t^i \equiv w_1 - c_t^i$, so that consumption in the second period is $c_{t+1}^{i2} = w_2 + \frac{p_t s_t^i}{p_{t+1}}$), and let s_t be the real savings per capita (per-young, of course). The market clearing condition in the money market is then $m_t = s_t$, because s_t represents, as well, the real money demand per capita in the economy. Then the following equation should, by definition, hold for $t \geq 1$: $s_t = \frac{1}{n} \sum_{i=1}^n s_t^i$.

3.2 Uncertainty Space and Information Structures

Even though there is no exogenous uncertainty so far in this set up, equilibrium variables like $(p_t)_{t=0}^\infty$ might well be stochastic processes, only by the fact that individuals might make arbitrary decisions, which affect those variables. Those decisions might depend on heterogeneous or even time varying rules and beliefs about future prices and other individuals actions. At this stage we could let Ω be the set of fundamental uncertainty in our economy, $\omega \in \Omega$ being interpreted as a sequence of all inflation rates and savings decisions in the economy (and everything that determines them), from period 1 up to infinity. Even without exogenous uncertainty, the absence of credit markets makes knowledge of future prices impossible, in principle, for any given individual. In fact, any individual can have different priors on future prices or on the way other agents make their decisions, generating in that way priors on those future prices.

We now introduce the model of *family participation*. So far, we could have thought of the economy as formed by a group of families, each of which had a member participating in every period. I will assume that, more generally, there is a fixed population N of families in the economy, but not all of them have its member participating every period. Only n families, out of the N available, have a member being born and participating in the economy as young agents at each period. In principle, a family can have two members in a given period participating in the economy, one as young and one as old; but this might not happen, and so we need to assume $N \geq n$. At the beginning of every period, n families are selected randomly by nature from the N available ones, so that each family in the n selected ones has an offspring that will participate as young in that period³. Ω will then include, in addition to what was said above, the information of which n families are selected each period.

Let $(\tau_r^i)_{r=1}^\infty$ be the stopping times process that governs family's i turns to have an offspring, for $i = 1, 2, \dots, N$. So, for example, for i and r given, τ_r^i is the random variable that determines when the r 'th turn of a member of family i to participate occurs. One condition for these variables is, then, that $\tau_r^i \leq \tau_m^i$ for all $r, m = 1, 2, \dots$ such that $r < m$.

Let now $(r_t^i)_{t=1}^\infty$ be the counting process that accumulates the number of turns of family i , so that r_t^i says the number of times that a member of family i has participated in the economy until and including time t . Let us define the following indicator function: for all $A \subset \Omega$,

$$\mathbf{1}_A(\omega) \equiv \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{otherwise} \end{cases}$$

For $A = \{\omega' : \tau_r^i(\omega') \leq t\}$, if $\omega \in \Omega$ is the true state of the world, and $\mathbf{1}_A(\omega) = 1$, that means

³An additional assumption, made in [15] for the experimental design is that families get assigned, when they participate, a random discount factor β^k in a set $(\beta^1, \beta^2, \dots, \beta^n)$. In order to guarantee the existence of perfect foresight equilibria, it is assumed that all the discount factors in this set are present in the ex-post utilities each period. It turns out that the cases in which β^k in randomly assigned produces the same kind of behavior in the experiments as the cases where it is constant. We can then concentrate in the latter case, which is cleaner for our analysis. In example 1 below this assumption is relaxed to include all experimental cases, and we illustrate the consequences.

that at t agent i has already had, or is currently having, her turn number r . We then have that

$$r_t^i = \sum_{r=1}^{\infty} \mathbf{1}_{\{\tau_r^i \leq t\}}$$

Our setup, including the stopping times and counting processes, is a general framework in which we can put the experimental design of [15]. In particular, it is imposed that $\tau_r^i < \tau_{r+2}^i$, for $i = 1, 2, \dots, N$, $r = 1, 2, \dots$. This is so because if an agent participates at t , he can not begin participating again at $t + 1$, when he is already present as an old agent, or at $t + 2$, because he is not allowed to live two lives in a row. It is also required that agent's turns to participate be independent over time and across agents. In the experiments, only each respective agent has information about their own turns to play, their discount factors, and their individual decisions. With this in place, I want to explore the use of private experience as a possible explanation of the features in the experimental data not explained by the models with rules that use only public information.

Each family has some information on past events occurred on Ω : \mathcal{F}_t^i is the sigma algebra that formalizes the information available to family i of generation t , for all families and every t . Let $(P_t^1, \dots, P_t^n)_{t=1}^{\infty}$ be the probability measures or *beliefs* by each individual on Ω . (They might or might not be updated using Bayes rule.) Let now $\mathcal{F} \equiv \sigma < \bigvee \mathcal{F}_t^i >$. Then $(\Omega, \mathcal{F}, ((\mathcal{F}_t^1, P_t^1), \dots, (\mathcal{F}_t^n, P_t^n))_{t=1}^{\infty})$ is the space that summarizes the uncertainty, information structure and agents' beliefs in our economy. We will assume that P is the probability distribution on Ω that governs family participation, and that P_t^i coincides with P in that dimension, for all i .

Regarding information, we will say that $\{\tau_r^i \leq t\} \in \mathcal{F}_t^i$ for all $r, t = 1, 2, \dots$, $i = 1, 2, \dots, N$. The fact that, for a given time t , $\{\tau_r^i \leq t\} \in \mathcal{F}_t^i$ holds for every r means that at each period of time t , family (or agent) i knows all the turns that it has had up to then; in other words, it knows at t if its r 'th turn has occurred or not, for every r . We will also say, more formally, that even if they do not use it, families are able to observe all past prices and know when they have participated so far at each date:

Assumption 1 (FST) For all i, s ,

$$\mathcal{F}_t^i = \sigma < r_s^i, \pi_s, s = 1, 2, \dots, t > .$$

Since this feature eliminates from the model all uncertainty that comes from (public) information, we could regard $(\Omega, \mathcal{F}, ((\mathcal{F}_t^1, P_t^1), \dots, (\mathcal{F}_t^n, P_t^n))_{t=1}^{\infty})$ as a space of *Subjective Uncertainty*. By this we mean that if there is a stochastic character of the equilibrium variables, it has to come from individual subjective beliefs and/or actions.

Regarding preferences, I will always assume von Newman–Morgenstern utility functions for the agents in the models presented, taking for granted that the actual experimental agents I want to model effectively have those kind of utilities, although they might not act as maximizers. So I will

always assume that agent i of generation t , for $i \in \{1, 2, \dots, N\}$, has *ex-ante* utility function of the form⁴

$$E^{i,t} \left\{ \left[\ln(w_1 - s_t^i) + \beta^i \ln(w_2 + \frac{s_t^i}{\pi_{t+1}}) \right] \parallel \mathcal{F}_t^i \right\}.$$

An economic environment, ε , is a vector

$$\varepsilon = \left(h_{-1}, d, w_1, w_2, \beta^1, \dots, \beta^n, ((\mathcal{F}_t^1, P_t^1), \dots, (\mathcal{F}_t^n, P_t^n))_{t=1}^\infty \right),$$

where $w_2 \ll w_1$, $0 < w_1 < d$, and, for $i \in \{1, 2, \dots, N\}$, $\beta^i \in \mathbf{R}_+$, and (\mathcal{F}_t^i, P_t^i) are related to Ω as described above.

3.3 Equilibrium Notions and Rational Behavior

With the participation setup in place aggregate savings at t is given by:

$$s_t = \frac{1}{n} \sum_{\substack{i \in \{1, 2, \dots, N\}: \\ r_t^i = r_{t-1}^i + 1}} s_t^i$$

The weakest form of equilibria we could think of, then, in this framework, are processes⁵ $(\pi_t, s_t)_{t=1}^\infty$ that satisfy, for all $i \in \{1, 2, \dots, N\}$, $s_t = \frac{s_{t-1}}{\pi_t} + d$, and $0 \leq s_t^i \leq w_1$. This definition has built in, as we wanted, budget constraints (3.3), equation (3.3) implies money market clearing and government balanced budget. The first inequality in (3.3) guarantees that $w_2 + s_t^i/\pi_{t+1}$ is bigger than zero. Similarly, the second inequality in (3.3) implies that the first period consumption of the same agent ($c_t^{i1} = w_1 - s_t^i$) is positive. Notice that nothing is said about the way in which s_t^i is decided by the participating agents. The next sequence of definitions will help us to put in contrast different assumptions in this regard, in our search for a reasonably good match between models and actual behavior. Savings decisions by individuals in fact depend in some way, which we would like to be able to identify, by some function f^i for individual i of generation t , on beliefs, information and own parameters. Equations (3.3) and (3.3) would imply,

⁴In terms of consumption goods, this means that, for any $\omega \in \Omega$, and $\forall \omega' \in \Omega$,

$$\begin{aligned} E^{i,t} [\ln(c_t^{i1}) + \beta^i \ln(c_{t+1}^{i2}) \parallel \mathcal{F}_t^i](\omega') &= \int_{\Omega} [\ln(c_t^{i1}(\omega)) + \beta^i \ln(c_{t+1}^{i2}(\omega))] P_t^i[d\omega \parallel \mathcal{F}_t^i](\omega') \\ &= \ln(c_t^{i1}(\omega')) + \beta^i \int_{\Omega} \ln(c_{t+1}^{i2}(\omega)) P_t^i[d\omega \parallel \mathcal{F}_t^i](\omega'), \end{aligned}$$

where $P_t^i[\cdot \parallel \mathcal{F}_t^i](\omega')$ is the conditional probability distribution of ω given \mathcal{F}_t^i , at ω' . c_t^{i1} is known at t , so it is measurable with respect to \mathcal{F}_t^i .

⁵This definition is made in terms of $(\pi_t)_{t=1}^\infty$. To transform it into a definition in terms of $(p_t)_{t=0}^\infty$, we need to assume an initial p_0 . Now, since markets clear at time $t = 0$, $m_0 = s_0$; also $m_0 = h_0/p_0$, $h_0 = h_{-1} + p_0 d$, where h_{-1} is the transfer of money that the government makes to each initial old person, and $s_0 = \frac{1}{n} \sum_{i=1}^n s_0^i$, so that $h_{-1}/p_0 + d = \frac{1}{n} \sum_{i=1}^n s_0^i$. It follows that the initial condition is in fact h_{-1} , because p_0 is determined by $p_0 = \frac{h_{-1}}{(s_0 - d)}$.

then, that the inflation process $(\pi_t)_{t=1}^\infty$ depends, ultimately, on these beliefs, information, own and also aggregate parameters.

Equilibria in which agents are maximizers would have to satisfy the first order condition for maximization:

$$-1 \frac{1}{w_1 - s_t^i + E^{i,t} \left\{ \left[\frac{\beta^i}{\pi_{t+1} w_2 + s_t^i} \right] \parallel \mathcal{F}_t^i \right\}} = 0, \text{ for all } i.$$

An additional level of rationality would be present in an equilibrium if agents updated their beliefs using Bayes rule, formally, if for all i , and $A \in \mathcal{F}$, P_t^i is a random variable measurable with respect to $\sigma < \bigvee_{s \leq t} \mathcal{F}_s^i$ and

$$P_t^i(A)(\omega) = P_0^i \left(A \parallel \sigma < \bigvee_{s \leq t} \mathcal{F}_s^i \right) (\omega), \quad \forall \omega \in \Omega.$$

The most commonly held assumption on studied equilibria require an even stronger level of rationality, the *Rational Expectations* assumption. A Rational Expectations Equilibrium (*REE*) would require, in addition to the above, that for each i ,

$$E^{i,t} \left\{ \left[\frac{\beta^i}{\pi_{t+1} w_2 + s_t^i} \right] \parallel \mathcal{F}_t^i \right\} = \frac{\beta^i}{\pi_{t+1} w_2 + s_t^i}. \text{ This requirement implies that agents assign probability one to the inflation}$$

is a *REE*, then, for all i and $s_t^i = \frac{\beta^i w_1 - \pi_{t+1} w_2}{1 + \beta^i}$, which follows from equations (3.3) and (3.3), by definition of a *REE*. $\alpha w_1 - \gamma w_2 \pi_{t+1}$, where $\frac{1}{n} \sum_{i=1}^n \frac{\beta^i}{1 + \beta^i}$ and $\frac{1}{n} \sum_{i=1}^n \frac{1}{1 + \beta^i}$. From (3.3), we get that $(\pi_t)_{t=1}^\infty$ then satisfies $\pi_{t+1} = (c + 1) - \frac{b}{\pi_t}$, where $b \equiv \frac{\alpha w_1}{\gamma w_2}$ and $c \equiv b - \frac{d}{\gamma w_2}$. The difference equation (3.3) for $(\pi_t)_{t=1}^\infty$ in a *REE* is represented in Figure 1.

Depending on the initial π_t , there are uncountably many rational expectations equilibria that satisfy (3.3), under suitable conditions. There are two rational expectations equilibria, *LSS* and *HSS*, called *steady state*, or *stationary*, in which the elements of $(\pi_t)_{t=1}^\infty$ are constant. Let one of the constants be π^L , and the other π^H , with $\pi^L < \pi^H$. The rest of rational expectations equilibria are called *non-stationary*, and have the particularity that if for any of them $\pi_1 \in (\pi^L, \infty)$, then $\pi_t \rightarrow \pi^H$. This is why the equilibrium *HSS* is called *stable*. The equilibrium *LSS*, on the other hand, is not only *not* stable in that sense, but also has the property that if, for any equilibrium $(\pi_t)_{t=1}^\infty$, $\pi_1 \neq \pi^L$, then $\pi_t \neq \pi^L$ for all t .

An intriguing aspect of the rational expectations equilibria is that an increase in d , the per-young agent real government deficit, raises, on the one hand, π^L , but lowers, on the other, π^H , and vice-versa. (We can picture this by translating the graph in Figure 1 downwards when d increases.) So that the prescribed *classical* policy of lowering money financed government deficit seeking reduction of the inflation rate would “work” in the present setup only if the economy is in the low inflation equilibrium, which is the unstable one. On the other hand, that classical policy of lowering the deficit would in fact increase inflation in the long run if the economy is in any other equilibria, which are, in some loose sense, overwhelmingly more probable to occur. I put *work* in quotation marks because once d has been changed to $d' < d$, say, the economy might not jump right away to the new, lower *LSS*(d'). If the initial inflation for the new *REE* is $\pi^L(d) \neq \pi^L(d')$, the new process will converge to *HSS*(d'), and inflation will *rise*, not go down, in the long run, as

⁶ $\pi^L = 2$ and $\pi^H = 3.5$ in Figure 1.

Figure 1: Graph of the Difference Equation Representing *REE*

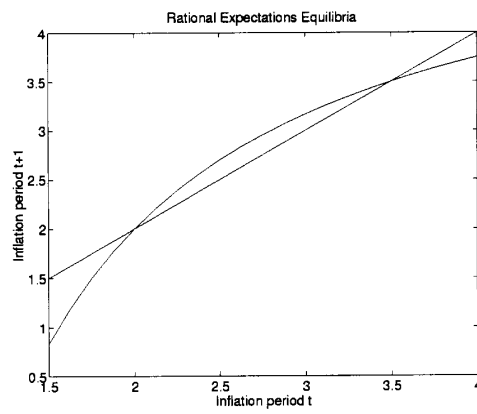


Figure 1:

can be imagined by looking at Figure 1. So that we hardly have a guarantee that the policy would ever work, even if the economy is initially at a *LSS*.

It is because of econometric evidence regarding the relationship between government deficit and inflation, that [12] suggested to look at different kinds of equilibria, the ones in which the classical policy really works, in order to fit the data. It was suggested there the kind of equilibria in which agents use decision rules in such a way that, in the long run, they learned both, to behave *rationally* (as if they had *-right-* beliefs about future inflation rates, and they performed maximization of their utility functions), and also to stay or be attracted to the rational expectations equilibrium in which the classical policy works. It turns out that adaptive behavior produces such kind of equilibria, and one example is the Least Squares model proposed by [13], and used in [15] to explain the experimental data. This model has the same steady states as the *REE* model⁷, but the *LSS* is (locally) stable in this case. So small perturbations in d will produce the desired effect on long run inflation. Although agents are not rational at any given period if the economy does not start at *LSS*, people learn to be so as time goes on. The problem with such a model is that the implied time paths of inflation are very smooth and monotonic, unlike the ones in the experiments⁸.

We are led from our foregoing discussion to think that, as we have defined it, rationality assumptions on agents behavior do not seem to explain observed behavior regarding the limit of a learning process, as suggested in [12] and observed in experimental data. The alternative kind of behavioral rules we are led now to explore, called Adaptive learning rules, imply that agents are *boundedly rational* in the sense that they do not use the Bayesian rule to update their beliefs, even though they can be, in some cases at least, maximizers. Now, the kind of adaptive rule proposed in Marimon–Sunder to explain the data does not seem enough to explain other aspects present in the decision making of actual experimental subjects, other than the long run tendency. For that purpose I use more general versions of that kind of rule, as described below.

⁷ As we will see, in the Least Squares model used in [15], every agent makes the same price estimation and behave as a maximizer. So instead of the actual inflation rate π_{t+1} in equation (3.3) we have π_{t+1}^e . Using (3.1) we get $s_t = \alpha w_1 - \gamma w_2 \pi_{t+1}^e$. From (3.3) and assuming $\pi_{t+1}^e = \pi_t^e = \pi_{t+1}$ (in order to see if there are *REE* steady states), we get the same characteristic equation that comes from (3.3).

⁸ It is worthwhile to point out that a Bayesian kind of behavior could also produce a learning process that would result in convergence to a *REE*. In this regard, I want to point out that [8], [9] have a result, similar to that of [5] that links Bayesian equilibria with rational expectations equilibria through Bayesian learning. Although the adaptation of their result to our framework would be made for finite and constant range for π_t , which is an inadmissible restriction in our model, we conjecture that it applies, even when $\pi_t \in \mathbf{R}_+$ instead. In that case people would learn both to be and to believe in some rational expectations equilibrium if they are maximizers, do Bayesian updating, and also have “a grain of truth” in their initial beliefs. Now, the specific *REE* to which the process would converge depends on initial conditions and initial beliefs. This means that Bayesian learning leads agents to a *REE*, but not necessarily (and very improbably indeed, since this is but a single one among uncountably many possibilities), to the classical *LSS*(ε). Although Bayesian learning is useful in other contexts, in ours it doesn't seem appropriate.

3.4 Adaptive Learning with Private Experience

The general kind of rules I am going to use in this paper as possible explanations of real behavior are instances, as suggested above, of *adaptive* rules, mainly *Robbins-Monro* type of algorithm (RMA). They are described in the following definition. Here $\hat{\theta}_t^i$ is meant to represent the group of parameters or the function, depending on the application, being estimated and learned about by agent i , and f^i is his decision rule. Finally, γ_t and M will be meant to be the learning rate and learning rule, respectively, used by agents in period t .

Definition 1 *Given the stopping times process $(\tau_r^i)_{r=1}^\infty$, $i = 1, 2, \dots, N$ and an economic environment $\varepsilon = (h_{-1}, d, w_1, w_2, \beta^1, \dots, \beta^n, ((\mathcal{F}_t^1, P_t^1), \dots, (\mathcal{F}_t^n, P_t^n))_{t=1}^\infty)$, and given $\hat{\theta}_1^i$, $i = 1, 2, \dots, N$, the process $(\pi_t, (s_t^i)_{i=1}^N, s_t)_{t=1}^\infty$ is an **Adaptive Equilibrium**, **AE**(ε), if the following equations hold for all i :*

$$\hat{\theta}_{t+1}^i = \hat{\theta}_t^i + M(\tau_t^i, \pi_t, \hat{\theta}_t^i) s_{t+1}^i = f^i(\hat{\theta}_{t+1}^i) s_{t+1}^i = \frac{1}{n} \sum_{\substack{i \in \{1, 2, \dots, N\}: \\ r_{t+1}^i = r_t^i + 1}} s_{t+1}^i \pi_{t+1} = \frac{s_t}{s_{t+1} - d}, \text{ where } \pi^t \equiv (\pi_1, \pi_2, \dots, \pi_t), \text{ and functions } M : \mathbf{R}^3 \rightarrow \mathbf{R}, \text{ and } f^i : \mathbf{R} \rightarrow [0, w_1], \text{ for all } i.$$

As it will become clear in the examples, these kind of learning rules correspond to a certain kind of myopic beliefs, and, in a loose sense, boundedly rational behavior by the agents. Notice that even though a family might not be participating, it might still be estimating the savings it would decide if it were participating. Also notice that I am focusing in the case where M and γ_t are common across agents, which means that all agents use the same learning algorithm. Finally, f^i are constant over time. These features of our alternative model are enough to produce what we want.

In general, $(\pi_t, s_t)_{t=1}^\infty$ in an *AE* is a stochastic vector process. Nevertheless, the following result clarifies the exact conditions when it is not. Before that, I will clarify that, when I say that $\sum_{\substack{i \in \{1, 2, \dots, N\}: \\ r_{t+1}^i = r_t^i + 1}} f^i(\hat{\theta}_{t+1}^i)$ depends only on $(\hat{\theta}_{t+1}^i)_{\substack{i \in \{1, 2, \dots, N\}: \\ r_{t+1}^i = r_t^i + 1}}$, I mean that there exist functions (g^1, g^2, \dots, g^N) such that $\forall \omega \in \Omega$,

$$\sum_{\substack{i \in \{1, 2, \dots, N\}: \\ r_{t+1}^i(\omega) = r_t^i(\omega) + 1}} f^i(\hat{\theta}_{t+1}^i(\omega)) = \sum_{j=1}^n g^j(\hat{\theta}_{t+1}^{k_j}(\omega)),$$

where $(\hat{\theta}_{t+1}^{k_1}, \hat{\theta}_{t+1}^{k_2}, \dots, \hat{\theta}_{t+1}^{k_n})$ is some permutation of $(\hat{\theta}_{t+1}^i)_{\substack{i \in \{1, 2, \dots, N\}: \\ r_{t+1}^i(\omega) = r_t^i(\omega) + 1}}$. This assumption might be natural when there is a large population of participants and the distribution of decisions is time invariant. Otherwise it is a very strong (and unnatural) assumption, but it turns out that it is crucially implicit in the version of the *OLS* model in [15], and in every representative agent model.

When that assumption holds, we can then well define the function $G : \mathbf{R}^n \rightarrow \mathbf{R}_+$ as

$$G(\hat{\theta}_{t+1}^{k_1}, \hat{\theta}_{t+1}^{k_2}, \dots, \hat{\theta}_{t+1}^{k_n}) \equiv \sum_{j=1}^n g^j(\hat{\theta}_{t+1}^{k_j}).$$

Similarly, when I say that $(\tau_{t+1}^i, \hat{\theta}_t^i, \pi_t) \rightarrow M(\tau_{t+1}^i, \hat{\theta}_t^i, \pi_t)$ does not depend directly on τ_{t+1}^i , I mean simply that $M(\tau_{t+1}^i, \hat{\theta}_t^i, \pi_t) = \tilde{M}(\hat{\theta}_t^i, \pi_t)$, for a well defined function \tilde{M} .

Assumption 2 (EST) $(\tau_{t+1}^i, \hat{\theta}_t^i, \pi_t) \rightarrow M(\tau_{t+1}^i, \hat{\theta}_t^i, \pi_t)$ does not depend directly on τ_{t+1}^i , for all $i \in \{1, 2, \dots, N\}$.

Assumption 3 (UNI) 1. $\hat{\theta}_1^i = \hat{\theta}_1^j$, $i, j = 1, 2, \dots, N$,

$$2. \sum_{\substack{i \in \{1, 2, \dots, N\}: \\ r_{t+1}^i = r_t^i + 1}} f^i(\hat{\theta}_{t+1}^i) \text{ depends only on } \left(\hat{\theta}_{t+1}^i \right)_{\substack{i \in \{1, 2, \dots, N\}: \\ r_{t+1}^i = r_t^i + 1}}.$$

The following proposition will allow us to carefully separate the different sources of randomness in our model, so that we can study their meaning:

Proposition 1 Let $(\pi_t, (s_t^i)_{i=1}^N, s_t)_{t=1}^\infty \in AE()$. Then, for arbitrary non deterministic stopping times $(\tau_r)_{r=1}^\infty$, $(\pi_t, s_t)_{t=2}^\infty$ is deterministic if and only if assumptions EST and UNI hold. Furthermore, under these conditions, $\hat{\theta}_t^i = \hat{\theta}_t^j$, $i, j = 1, 2, \dots, N$.

Proof:

I will sketch here the *only if* part. Suppose that (s_t) is deterministic and assume conditions UNI hold, but EST does not hold. Then, different estimations $\hat{\theta}_{t+1}^i$ would result from different stopping times (different experience). Since the aggregate s_{t+1} depends on the (random) selection, it will be random, a contradiction. This shows that we can not have deterministic (s_t) with private experience use, even though first estimations are common and we have the *aggregation uniformity* condition UNI 2. Now if only UNI 1. does not hold, the estimators $\hat{\theta}_{t+1}^i$ will, again, be different, and time varying, since previous estimators have been different (in spite of using the same public information available). Since the aggregate s_{t+1} depends on the selection of families, it will be random, a contradiction again. The case where UNI 2. alone is violated only matters if $f^i \neq f^j$ for at least a pair i, j , since otherwise it is trivial. Even though $\hat{\theta}_{t+1}^i = \hat{\theta}_{t+1}^j$, for all i, j , we have that the aggregate s_{t+1} depends on the group of families selected, which have different decision rules. s_{t+1} is then random (with a finite range in this case), a contradiction, *Q.E.D.*⁹

Proposition 1 is a negative result: it shows that extremely strong assumptions should be met in order to eliminate randomness in the theoretical paths of the aggregate variables. In the same loose sense expressed before for the probability of occurrence of *LSS* in a *RRE*, it is extremely improbable that these conditions are met in any real world situation. Nevertheless, as we will see

⁹ The rigorous, complete proof is available upon request.

below, these are precisely the assumptions implicitly made in the ubiquitous representative agent learning models. In a positive way, we can interpret this result as saying that in an environment with stochastic turns, if we impose the necessary uniformity on the players at the aggregate level, the only feature that can produce stochastic paths for savings and inflation rates in this economy, is that agents *do not* use only public information in their decision rules¹⁰. This idea is made precise in the following

Corollary 2 *Let $(\pi_t, (s_t^i)_{i=1}^N, s_t)_{t=1}^\infty \in AE()$. Assume that, for arbitrary stochastic stopping times $(\tau_r)_{r=1}^\infty$, conditions 2 in proposition 1 hold. Then, if $(\tau_t^i, \hat{\theta}_t, \pi_t) \rightarrow M(\tau_t^i, \hat{\theta}_t, \pi_t) \in \mathbf{R}$ depends directly on τ_t^i for at least one $i \in \{1, 2, \dots, N\}$, in at least some period $t \in \{1, 2, \dots\}$, $(\pi_t, s_t)_{t=1}^\infty$ is a (non-deterministic) stochastic vector process.*

Notice that condition UNI in proposition 1 is satisfied if the initial conditions are the same for all the players, and they all have the same decision rule: $f^i = f^j$ for all $i, j = 1, 2, \dots, N$. So that our stopping times framework does not rely in heterogeneity of the mechanisms the players use to take decisions. Now, the opposite can happen: if agents use only public information on their decision rules, and the uniformity at the aggregate level condition is not met, then savings and inflation in this economy will be stochastic processes. This time, randomness will depend in heterogeneity of decision rules, reflected at the aggregate level. This is the randomness that could be eliminated automatically if we imposed some kind of constant distribution of many participants. More formally,

Corollary 3 *Let $(\pi_t, (s_t^i)_{i=1}^N, s_t)_{t=1}^\infty \in AE()$. Assume that, for arbitrary stochastic stopping times $(\tau_r)_{r=1}^\infty$, condition EST in proposition 1 hold. Then, if at least for a pair i, j , $i, j \in \{1, 2, \dots, N\}$ $\hat{\theta}_1^i \neq \hat{\theta}_1^j$, and/or*

$$\sum_{\substack{i \in \{1, 2, \dots, N\} : \\ r_t^i = r_{t-1}^i + 1}} f^i(\hat{\theta}_{t+1}^i)$$

depends on $\{i \in \{1, 2, \dots, N\} : r_t^i = r_{t-1}^i + 1\}$ for at least one period $t \in \{1, 2, \dots\}$, $(\pi_t, s_t)_{t=1}^\infty$ is a (non-deterministic) stochastic vector process.

Of course, the process will have more randomness, talking in a loose sense, the more individuals and/or time periods are involved in the conditions on corollaries 1 and 2, in general. Our discussion puts us in a position to see why representative agent learning models, like the ones in [15] are not appropriate to capture the specific feature of randomness involved in the experiments. If a single learning algorithm is applied to explain the aggregate data, as in a representative agent model, the group of assumptions UNI in proposition 1 are effectively guaranteed to be met. If, on top of that, the representative agent uses only public information, like inflation rates, in his learning

¹⁰ Strictly speaking, erratic oscillations are not ruled out by deterministic paths, for the path can be deterministic but *chaotic*. Chaotic time paths in the perfect foresight case are ruled out by our utility functions. They are ruled out as well in our examples of the adaptive case by our rules with homogeneous agents, to which the model converges. Chaotic paths are ruled out by any algorithm that produces learning, of course.

algorithm¹¹, condition EST of proposition 1 is met. It follows that, other than the randomness inherent in the initial period, the *theoretical* processes generated by the algorithm are deterministic, and the paths of its elements are smooth (other than the chaotic case, which is not present in our functional forms), as our simulations exemplify.

We can ask: is $(\pi_t, s_t)_{t=1}^\infty$ deterministic if people use public information, even though the $\hat{\theta}_1^i$'s are random? We could think that the randomness of the initial conditions could determine a stochastic path. This is in fact the case, except when people participate in every period, as we will see next. The reason is because when people participate at different times, they accumulate different experiences, and, since they use them in their decision, the initial randomness is translated to following periods via the aggregation of time varying components. Since the time paths of $(\pi_t, s_t)_{t=1}^\infty$ in the experiments do not look deterministic, we are able to conjecture that if the aggregation condition of the proposition is somehow guaranteed by the experimental design, agents in fact use, at least in part, private information when they make decisions. But this can only hold in the case when different rates of accumulation of experience (allowed by the presence of random participation times) are present among the agents. In fact, as a complement to proposition 1 we have¹²:

Proposition 4 *Let $(\pi_t, (s_t^i)_{i=1}^N, s_t)_{t=1}^\infty \in AE()$. Let $\tau_r(\omega) = r \quad \forall \omega \in \Omega, r = 1, 2, \dots$. Then $(\pi_t, s_t)_{t=2}^\infty$ is deterministic.*

This result means that the fact of people using private experience and different (although constant over time) learning rules does not, by itself, guarantee randomness in the time paths of aggregate variables. The time paths of those variables are in fact smooth, in spite of different (*random*) initial decisions, different decision rules, and use of private experience. So this kind of heterogeneity does not guarantee our objective, leading us to clearly appreciate the fact that different levels of experience accumulated (made possible by different participation times in this context) are essential to produce the desired erratic oscillations.

4 Examples

To illustrate our model, I will show some examples and simulations.

¹¹ The representative agent is an aggregate, and its components can have different experiences, as in conditions for corollary 1.

¹² The formal proof (available upon request) of this obvious fact, omitted here since it is only bookkeeping, disentangles the stochastic character of the *AE* in a revealing way, highlighting from a different perspective than proposition 1, the same intuitive generalization of the homogeneous case involved in our model.

4.1 Example 1: *OLS* Learning

The first example of an adaptive equilibrium is the correspondence *OLS*, in which inflation estimation is made by Recursive Ordinary Least Squares, and agents behavior is based on the belief that the estimation is certain to occur, for purposes of utility maximization. In fact, in an economy $\in \Sigma^{FST}$, let $\pi_{t+1}^{e,i}$ be the estimated inflation for $t+1$ by agent i at time t , based on previous values of π_t , and let $(\pi_{t+1}^{e,i}) = 1$ for all P_t^i in . If we fix $\omega \in \Omega$, we then have that $E^{i,t} \left\{ \left[\frac{\beta^i}{\pi_{t+1} w_2 + s_t^i} \right] \middle| \mathcal{F}_t^i \right\} (\omega) = \frac{\beta^i}{\pi_{t+1}^{e,i} w_2 + s_t^i(\omega)}$, since s_t^i is measurable with respect to \mathcal{F}_t^i .

If we assume that agents are maximizers, then the equilibria to be generated in this example will be maximizing equilibria. Of course, since $\pi_{t+1}^{e,i} \neq \pi_{t+1}$ in this case, in general, $(.) \neq P_0^i(.||)$, so that those equilibria can not be Bayesian. Agents are not rational, in the sense that they do not do Bayesian updating; in fact, families in future times will continue to act, in this example, as if they believed blindly in these specific kind of predictions, even though all previous family predictions might have been completely wrong. To satisfy (3.3) and (4.1), we need that, $s_t^i = \frac{\beta^i w_1 - \pi_{t+1}^{e,i} w_2}{1 + \beta^i} \cdot \pi_{t+1}^{e,i}$ is calculated in the following way: $p_{t+1} = \beta p_t$ is believed to be true, and β is estimated by¹³ $\hat{\beta}_{i,t+1} = \hat{\beta}_{i,t} + \frac{1}{(t+1)} \hat{R}_{t+1}^{-1} p_{t-1} (p_t - \hat{\beta}_{i,t} p_{t-1}) \hat{R}_{i,t+1} = \hat{R}_{i,t} + \frac{1}{(t+1)} \left((p_{t-1})^2 - \hat{R}_{i,t} \right)$, with $\hat{\beta}_{i,1} = \hat{R}_{i,1} = 0$, . Then, if $p_t^{e,i} \equiv \hat{\beta}_t p_{t-1}$, $\tilde{p}_{t+1}^{e,i} = \hat{\beta}_t p_t^{e,i}$ and $\pi_{t+1}^{e,i} \equiv \tilde{p}_{t+1}^{e,i} / p_t^{e,i}$, we have, in fact, that $\pi_{t+1}^{e,i} = \hat{\beta}_t$. Since at time t all the information on past prices is publicly available and $\hat{\beta}_{i,1} = \hat{R}_{i,1} = 0$, we have that $\hat{\beta}_{i,t} = \hat{\beta}_{j,t} \equiv \hat{\beta}_t$ and $\hat{R}_{i,t} = \hat{R}_{j,t} \equiv \hat{R}_t$, $i \neq j$, $i, j = 1, 2, \dots, n$, so that the value of $\pi_{t+1}^{e,i}$, defined to be $\pi_{t+1}^{e,i}$ and calculated by the above equations, is common across agents with a turn to play at time t . As in the experimental design, assume now that when the n families are selected, each of the elements of the vector $(\beta^1, \beta^2, \dots, \beta^n)$ of discount factors is assigned randomly to the different families of the selected group (of size n) in question. It then follows that the average savings at t , s_t is given by

$$\begin{aligned} s_t &= \frac{1}{n} \sum_{\substack{i \in \{1, 2, \dots, N\}: \\ r_t^i = r_{t-1}^i + 1}} s_t^i \\ &= \alpha w_1 - \gamma w_2 \pi_{t+1}^e, \end{aligned} \tag{1}$$

where α and γ are as defined in equations (3.3). Using (3.3) it then follows that the equilibria we seek will satisfy $\pi_t = \frac{\pi_t^e - b}{c - \pi_{t+1}^e}$ for all. For as described, *OLS()* is defined to be any process $(\pi_t, (s_t^i)_{i=1}^N, s_t)_{t=1}^\infty$ generated by equations (4.1)–(4.1). We have then, that any process $(\pi_t, (s_t^i)_{i=1}^N, s_t)_{t=1}^\infty$ generated

¹³This Recursive *OLS* formula is useful because it does not use values of p_s for $s < t-1$, but is equivalent to the *OLS* one

$$\hat{\beta}_{i,t+1} = \frac{\sum_{s=1}^t p_s p_{s-1}}{\sum_{s=0}^{t-1} (p_s)^2}.$$

in this example is an Adaptive Equilibrium, with $\hat{\theta}_1 = 0$,

$$\hat{\theta}_{t+1}^i = \left(\hat{\beta}_{i,t+1}, \hat{R}_{i,t+1} \right),$$

$$M(\tau_{t+1}^i, \pi^t, \hat{\theta}_t) = \frac{1}{t} \left[(\hat{R}_{t+1})^{-1} p_{t-1} (p_t - p_{t-1} \hat{\beta}_{i,t}), \left((p_{t-1})^2 - \hat{R}_{i,t} \right) \right] f^i(\hat{\theta}_{t+1}) = \frac{\beta^i w_1 - \hat{\beta} w_2}{1 + \beta^i}.$$

Although f^i is not common across agents, s_t depends only on $(\hat{\beta}_t, R_t)$ through π_{t+1}^e , by (4.1) and (1), based on common public information. Similarly, π_t depends on common public information, by (4.1), and so, any resulting process $(\pi_t, s_t)_{t=1}^\infty \in OLS()$ generated in this example is deterministic, since all conditions of Proposition 1 are satisfied, where $OLS()$ means the set of all such equilibrium paths given a specific economic environment. Any element in $OLS()$ depends crucially on the initial conditions $\pi_1, s_1, (s_1^i)_{i=1}^N$, because the rest of the (deterministic) path can be easily generated by the formulas above. With this remarks in place, we state the following result, directly adapted from [15] and [13], assuming the appropriate conditions for d, w_1, w_2, h_{-1} and $(\beta^1, \beta^2, \dots, \beta^n)$:

Proposition 5 *With P_t^i as described above, any process $(\pi_t, s_t)_{t=1}^\infty \in OLS()$ converges to $LSS()$.*

In other words, given any initial conditions (within a range), the process of inflation rates generated by agents that forecast those inflation rates by way of recursive least squares using all public information converges to the rational expectations equilibrium in which the classical monetary policy works. Notice that without the constant vector of discount factors assumption, there is no way that the process could converge to a constant; without that assumption, α and γ in (1) would be random, and the process (s_t) could only converge, at most, to a random vector that depends directly on different possible combinations of discount factors assigned, determined by who gets to play by the stopping times processes. This is why that assumption, which is the expression of part 2 of UNI for this example, is essential in the experimental design when the discount factors are assigned randomly, in order to be able to replicate the conditions of the benchmark OLG model without exogenous uncertainty. Notice also that the individual savings processes might follow stochastic-like paths. From this we see that in general, the assumption that we need, in order to preserve the model we want to emulate, is to insure that the aggregate savings has a constant functional form relative to what individuals get to participate in the different periods. That assumption, at the same time of preserving the OLG benchmark model, allows us to isolate the relevant assumption that generates random oscillations in our analytical model and in the experiments, which is, then, the dependency or not of the decision rules upon public information alone.

In Figure 2, panel A, I show the result of a simulation of the path of aggregate savings of a OLS equilibrium. The environment is as described in Experiment 2 of [15]. As it can be seen, the path converges to LSS and is very smooth and, after period 10, convergence is even monotonic. Particulars about the simulations are given in the Appendix.

If, alternatively, people used only the price information they *experienced* when they participated, to forecast inflation, we would have:

$$M(\tau_{t+1}^i, \pi^t, \hat{\theta}_t) = \mathbf{1}_{[\tau_{\tau_t^i+1}^i = t+1]} \frac{1}{r_t^i} \left[(\hat{R}_{i,t+1})^{-1} p_{\tau_t^i-1} (p_{\tau_t^i} - \hat{\beta}_{i,t} p_{\tau_t^i-1}), \left((p_{\tau_t^i-1})^2 - \hat{R}_{i,t} \right) \right]$$

Figure 2: Panels A and B, respectively

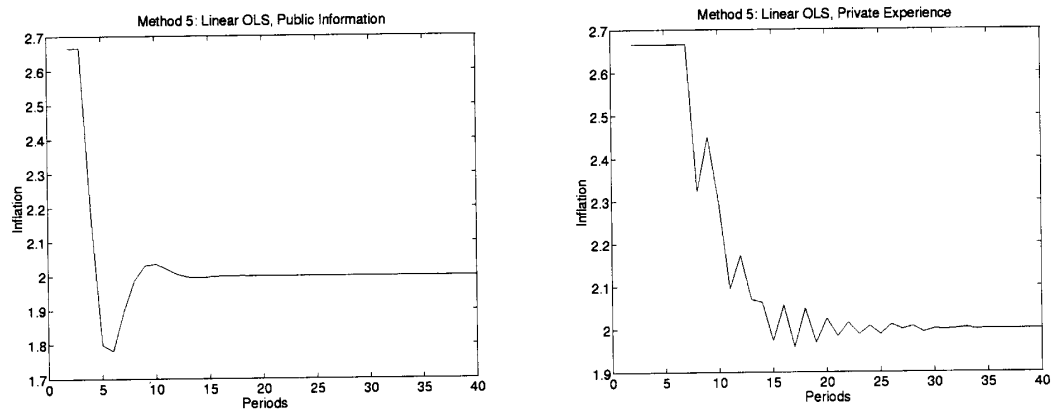


Figure 2:

Notice that here $\hat{\theta}_t^i$ (and s_t^i) is constant over periods in which family i does not participate. Even though the family is able to observe additional public information, it does not react to accommodate its estimators about the future unless a new turn occurs, and when that happens, the estimator takes into account only public information that occurred in previous turns. With this variation in the example, as M depends on τ_{t+1}^i , we have erratic time paths.

In Figure 2, panel B, I simulate an economy with the same stopping times as in panel A of the same figure. Here we have the variation to example 1 in which agents also use *OLS* to project prices, but use for that only the information on prices that occurred when they had turns to play in the past. This is an unnatural assumption, because in the experiments (and in real life, one could say) people have access to all history of inflation rates. Nevertheless, the simulation shows that if people really use *OLS*, and are maximizers, they have to use, at least in part, or some of them, that kind of information, in order for the paths of aggregate inflation to be erratic.

We then see that, although the characteristics of learning and equilibrium selection are captured by the *OLS* model, the feature of using only public information by the agents renders it incapable of reproducing the uncertainty, or erratic oscillations in the time paths typical of experimental data. The assumption that agents use *OLS* and only the information that they experience produce, at some extent, the desired erratic oscillations, as shown in panel B of Figure 2. But this assumption is unnatural for the method they are using, though, since agents observe all past information on prices. On the other hand, price forecasting is an intermediate (not so easy) step to do savings decisions. In addition, the graph does not match very well the experimental data, as can be seen at first glance. Below I present two more examples, in order of complexity required, in which agents make directly savings decisions. They improve in intuitive appeal, because of simplicity in the decision rules, and on effective similarity to the experimental time paths.

4.2 Example 2: Newton Approximation Algorithm

The second example of an adaptive equilibrium is the correspondence *NAA*, which stands for Stochastic Newton Approximation Algorithm, in which agents, although they are not rational or actual maximizers, try to estimate the rational savings choice that maximizes their utilities by way of using the alluded algorithm, and also the exclusive use of their own private experience. Here, when agent i has a turn to play at time $t + 1$, he tries to estimate not next period inflation rate, as in example 1, but rather the savings rate s_{t+1}^i such that

$$s_{t+1}^i \in \operatorname{argmax}_{(\tilde{s}_{t+1}^i)} E^{i,t+1} \left\{ \left[\ln(w_1 - \tilde{s}_{t+1}^i) + \beta_{t+2}^i \ln \left(w_2 + \frac{\tilde{s}_{t+1}^i}{\pi_{t+2}} \right) \right] \parallel \mathcal{F}_{t+1}^i \right\}.$$

To do so, he estimates s_{t+1}^i , the root that makes

$$-\frac{1}{w_1 - s_{t+1}^i} + E^{i,t+1} \left\{ \left[\frac{\beta_{t+2}^i}{\pi_{t+2} w_2 + s_{t+1}^i} \right] \parallel \mathcal{F}_{t+1}^i \right\} = 0$$

hold. A similar method was used by Woodford in [22]. Now, agent i 's beliefs about π_{t+2} are effectively that¹⁴ $P_{t+1}^i \left[\pi_{t+2} = \pi_{\tau_{r_t^i}^i + 1}, \beta_{t+2}^i = \beta_{\tau_{r_t^i}^i + 1}^i \right] = 1$. As in example 1, these beliefs will, in general, turn out to be $P_0^i(\cdot \| \mathcal{F}_{t+1}^i)$, and the equilibria to be generated here will not be Bayesian. Although this example shares non-rationality with example 1, here, in addition, agents are assumed not to be able to maximize their implied utility. In fact, when his turn arrives at time $t + 1$, agent

i tries to estimate the s_{t+1}^i that makes $U_1 \left(s_{t+1}^i, \pi_{(\tau_{r_t^i}^i + 1)} \right) \equiv -\frac{1}{w_1 - s_{t+1}^i} + \frac{\beta_{(\tau_{r_t^i}^i + 1)}^i}{\pi_{(\tau_{r_t^i}^i + 1)} w_2 + s_{t+1}^i} =$
hold. For this, he uses the Newton Approximation Algorithm $s_{t+1}^i = s_t^i - \gamma_{r_t^i} \left[U_{11} \left(s_t^i, \pi_{(\tau_{r_t^i}^i + 1)} \right) \right]^{-1} U_1 \left(s_t^i, \pi_{(\tau_{r_t^i}^i + 1)} \right)$
 is as in *RMA*. Effectively, we have, for each i , $s_{t+1}^i = s_t^i - \gamma_{r_t^i} \mathbf{1}_{[\tau_{r_t^i}^i + 1 = t+1]} \left[U_{11} \left(s_t^i, \pi_{(\tau_{r_t^i}^i + 1)} \right) \right]^{-1} \left[-\frac{1}{w_1 - s_t^i} + \frac{\beta_{(\tau_{r_t^i}^i + 1)}^i}{\pi_{(\tau_{r_t^i}^i + 1)} w_2 + s_t^i} \right]$

There is an economic intuition involved in the application of the Newton algorithm. People want to adjust in the right direction and amount savings decisions relative to their past experiences, given their beliefs. Let us see what the algorithm advises if, in his last turn to play, agent i saved *too much* relative to the actual inflation that occurred in the second period of that turn. U_1 is decreasing in savings at zero, and so U_1 is negative in that scenario; U_{11} is negative around the optimal s , in fact always negative in our case, so that $[U_{11}]^{-1} U_1$ is positive, and then, $-\gamma_t [U_{11}]^{-1} U_1$ is negative. The algorithm then advises the decision-maker to adjust downward the previous savings decision, which was too high. $-[U_{11}]^{-1} U_1$ says how much to adjust, which is optimal in some approximation sense, and γ_t scales further that adjustment. The opposite happens if the previous savings decision was *too low*.

Now we have that $s_{t+1} = \frac{1}{n} \sum_{\substack{i \in \{1, 2, \dots, N\} \\ r_{t+1}^i = r_t^i + 1}} \left\{ s_t^i - \gamma_{r_t^i} \left[U_{11} \left(s_t^i, \pi_{(\tau_{r_t^i}^i + 1)} \right) \right]^{-1} \left[-\frac{1}{w_1 - s_t^i} + \frac{\beta_{(\tau_{r_t^i}^i + 1)}^i}{\pi_{(\tau_{r_t^i}^i + 1)} w_2 + s_t^i} \right] \right\}$. We want, again $\frac{s_t(\omega)}{s_{t+1} - d}$. For each with P_t^i as above, define $NAA()$ to be the set of processes that satisfy (4.2)–(4.2). Then we have that with

$$\begin{aligned} \hat{\theta}_{t+1}^i &= s_{t+1}^i, \\ f^i(\hat{\theta}_{t+1}^i) &= \hat{\theta}_{t+1}^i, \\ M(r_{t+1}^i, \pi_t, \hat{\theta}_t^i) &= \mathbf{1}_{[\tau_{r_t^i}^i + 1 = t+1]} \gamma_{r_t^i} \left[U_{11} \left(s_t^i, \pi_{(\tau_{r_t^i}^i + 1)} \right) \right]^{-1} U_1 \left(s_t^i, \pi_{(\tau_{r_t^i}^i + 1)} \right), \end{aligned}$$

$NAA() \in AE()$.

¹⁴In words, this means that at $t + 1$, when family i is getting a turn to play ($\tau_{r_{t+1}^i}^i = t + 1$), it believes that next period's inflation, π_{t+2} , will be the same as the inflation that occurred during the following period to that when it last had a turn to play. That last turn was $\tau_{r_t^i}^i$, and the inflation that affected its savings then was, accordingly, $\pi_{\tau_{r_t^i}^i + 1}$. Similarly for the discount rate, in the case it is not constant. Here we are presenting directly the case where agents use private experience as more relevant.

In general, a process $(\pi_t, s_t)_{t=1}^{\infty} \in NAA()$ is stochastic because for any agent i $\hat{\theta}_{t+1}^i$ depends on τ_{t+1}^i , by proposition 1, although the aggregation assumption is guaranteed for that process by the fact that here $f^i = I$, the identity function, and if we still impose that each discount factor in $(\beta^1, \beta^2, \dots, \beta^n)$ gets assigned to each family. In addition to obtaining what we wanted in this example, erratic paths of the aggregate variables, and in spite of non-rational and non-maximizing behavior by the agents in this economy, our simulations show convergence to the right (*LSS*) limit, like in example 1.

In Figure 3, panels A and B, I simulate two examples of aggregate savings paths in *NAA* equilibria. The turns of participation are the same (although the initial conditions are different) as in Figure 2, where *OLS* equilibria are simulated, and the environment is, again, as in experiment 2 of [15]. We can observe the erratic oscillations in Figure 3, panels A and B, and the convergence to the *LSS* equilibrium, as in the experimental data. The difference between the two cases is the *learning rates*, the *gammas* of formula (4.2), as indicated in the respective panels. That difference suggests that the learning rates are actually a group of parameters to be considered to fit better the data using *calibration*, as in the business cycles literature¹⁵.

4.3 Example 3: Direct Savings Estimation

The purpose of this example, named Direct Savings Estimation (*DSE*), is to simplify the algebra complications of example 2. Here, agents need much less algebraic abilities in order to make decisions. As we will see in the simulations part, this simplification will not impinge upon the ability of the model to reproduce, very similarly to example 2, the features of the experimental evidence. Here, with the same beliefs about inflation rates as in example 2, agents also try to estimate directly the savings rate that maximizes their utility. But in this example the estimation is done in a much simpler way: savings are adjusted upward if the previous savings decision is considered to be smaller than the one that would have maximized utility, and downward if the opposite occurs. The algorithm would then be, at time $t + 1$: $s_{t+1}^i = s_{\tau_{t+1}^i - 1}^i + \mathbf{1}_{[\tau_{t+1}^i = t+1]} r_t^i (s_{\tau_{t+1}^i - 1}^{i,m} - s_{\tau_{t+1}^i - 1}^i)$, where $s_{\tau_{t+1}^i - 1}^{i,m}$ is the actual, *ex post*, maximizing savings of the previous participation. The rest is as in example 2¹⁶. The time paths are similar to

¹⁵ It is to notice that in this example and the next one, differently from the *OLS* one, different initial savings decisions alone could produce erratic time paths. But since simulations show in this case very fast convergence on the one hand, and the experimental data evidences more persistence than that in erratic behavior, on the other, we guess that more than different initial conditions is needed in these examples in order to match the data, if calibrations were to be done.

¹⁶ We could think that the agent might not be able to calculate now $s_{\tau_{t+1}^i - 1}^{i,m}$. He might, instead, figure something like: “the previous maximizing savings should have been about q_{t+1}^i ” where this number is a sort of unconscious guess, not an explicit calculation, given the memories of the actual payoff. The modeler could think of many ways to represent that guess. For example, in those circumstances, q_{t+1}^i could be equal to $(1 + \delta_{t+1}^i) s_{\tau_{t+1}^i - 1}^{i,m}$, where $\delta_{t+1}^i \in [-\delta^i, \delta^i]$, $\delta^i > 0$ is a small, (non random) number. In this case, the algorithm would be like the one above,

Figure 3: Panels A and B. Learning Rates 2 and 1, respectively

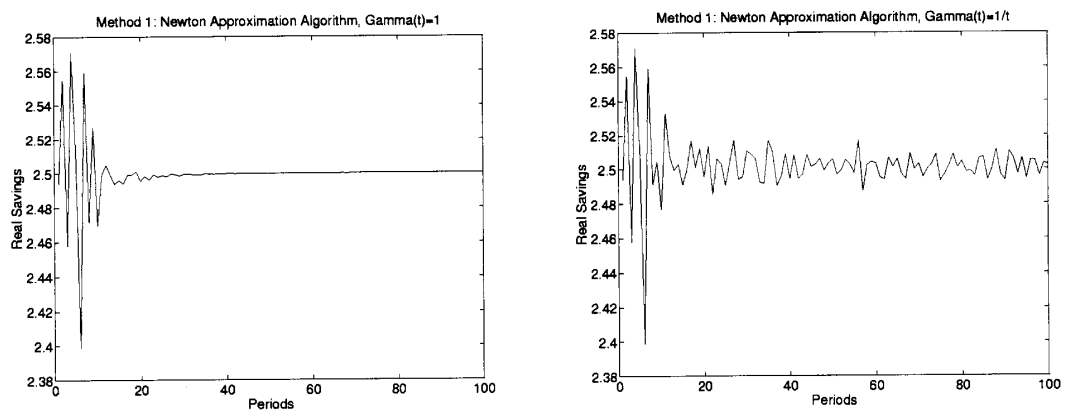


Figure 3:

Figure 4:

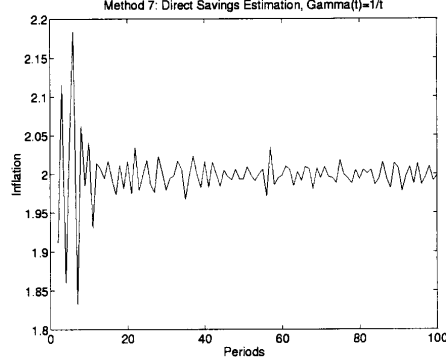


Figure 4 with Figure 3, panel B.

Figure 4:

those of example 2, as can be seen by comparing Figure 4 with Figure 3, panel B.

5 Final Remarks

For the sake of brevity and since it would take us away from the focus of the present paper, we do not deal with the issue of convergence of the AE processes. Standard techniques may be used to demonstrate that they converge to the (parametric) *LSS* limit. Chen and White ([6], [7]) have extended the analysis of convergence issues of Robins-Monro type of algorithm, like the AE, to the case of random (non-parametric) limits. In any case, our simulations provide examples of convergence of some AE processes to the *LSS* limit.

The time paths in the last two examples show in the simulations more oscillations, as seems to be the case in the experimental data, than the first one. Something that can be said, a priori, on the

with the guess instead of the actual ex post maximizing savings decision. This way of putting things would allow for the possibility of mixing the two causes of uncertainty (random guesses and random participation with private experience) that are actually present.

Figure 5: Panels A and B, respectively

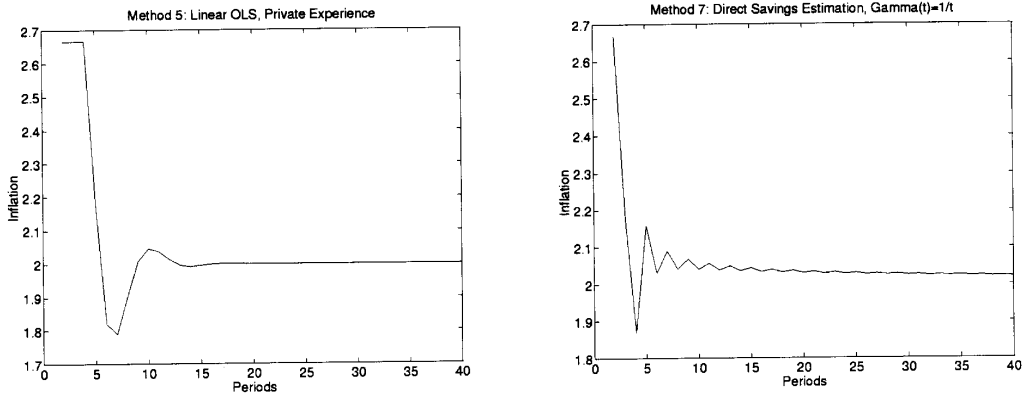


Figure 5:

possible performance of the different versions of the models, has to do with looking at them after isolating their *heterogeneity effect*. In fact, the effect that comes from different participation times, and the use of private experience, which translates into erratic time paths, tends to disappear as time goes on. This is due to the ongoing learning process¹⁷ (Of course, since there is a parametric limit, randomness also disappears together with heterogeneity.) Isolating the heterogeneity effect has to do, then, with looking at the specific model with homogeneous agents, and observing how the time paths behave in those cases. A *rich* model would show enough (deterministic) variability, in order to be able to reproduce certain frequencies in the data. A *poor* model, in contrast, would show only monotonic convergence.

The asymptotic properties of the difference equation that relate π_{t+1} and π_t in the examples remain to be analyzed. In the meantime, after observing many simulations, we can say that the *DSE* and *NAA* models display richer behavior than the *OLS* one. This can be seen, for example, if we compare panels A and B of Figure 5, where simulations of inflation time paths of the homogeneous-agents respective versions of those models are shown¹⁸. (The corresponding

¹⁷I take for granted, although I do not have the individual data to confirm it, that individuals in the actual experiments learn too, to make the *same* savings decision as the economy, as a whole, does.

¹⁸In fact, agents are not completely homogeneous here. They can be made homogeneous in the *OLS* case, but not in the other ones. The reason is because in the *DSE* case, for example, people can not participate every period, because in order for the newborns to take the payoff experience of their parents (which is used in these cases), they need current price information, which is not yet available. In order to overcome this problem I put the same group

Figure 6:

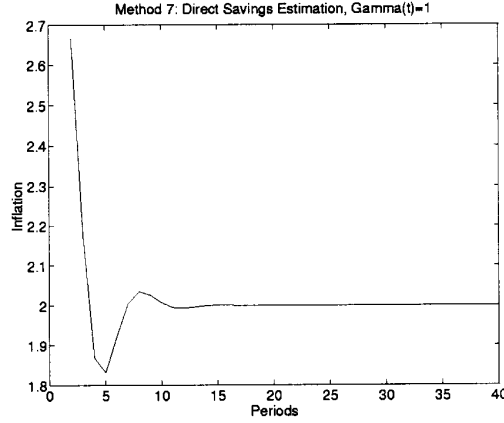


Figure 6:

graph for the *NAA* model is almost identical to the one for the *DSE* model.) While one after another simulation of the (homogeneous version of the) *OLS* model exhibited a pattern similar to that of panel A of Figure 5, basically all simulations of the *DSE* model with $\gamma_t = 1/t$ exhibited, on the contrary, an oscillatory pattern. On the other hand, the *DSE* and *NAA* models with $\gamma_t = 1 \forall t$ behave in a very similar way as the Linear *OLS* model, as can be seen, for the case of the *DSE* method, in Figure 6, with the same parameters other than γ_t . This shows that these last two models are able to reproduce basically all the cases, if the proper learning rates are set in place.

The effects of the kind of heterogeneity to which I appeal here in order to explain transitional dynamics might well disappear if the number of agents approaches infinity and is distributed in a suitable way. At the aggregate level, we can end up in this framework with a *rational* kind of behavior of the whole economy even in transitional periods in which the individual agents themselves are learning to behave rationally. (They would be learning to make the same decision that, on average, all the agents together are making *in every period*!) It would be interesting to find out whether that actually occurs in experiments where the number of subjects tends to be large.

of people to participate every other period, for every model. This is the closest approximation to homogeneity we can make for the *SDE* and *NAA* methods, and so, for comparability, I set the participation times to be odd-even in the *OLS* model as well. It turns out that the time paths with odd-even participation have the same general shape in this latter model as in the paths with completely homogeneous agents, anyway. This can be seen by comparing panels A of Figure 5, and Figure 2, where the same ω is used, but really homogeneous agents (in the sense that they use all the -common- information) participate.

For the case of finite and relatively small number of experimental subjects, our model seems to capture the main features observed in transitional periods, keeping assumptions on agent behavior close to the experimental design.

It is the case that the erratic oscillations that result from the present framework can be obtained in an equivalent form using some kind of idiosyncratic shocks around a homogeneous decision, or simply by assuming that the rules f^i depended on time and are stochastic. The interesting feature of my model would be to explain in a simple and intuitively appealing way where those shocks could come from other than from experimentation, and to illustrate how would we be able to handle a variety of cases in order to compare with the data¹⁹. It is like the general equilibrium models used to match aggregate data of the economy: in equilibrium, the resulting model is equivalent to a (commonly linear) econometric model that could be set up *ad hoc*. The interest of such a model would come from its parameters, and its interpretation, which comes from its underlying model (preferences, technology, decision making variables, and so on).

6 Conclusion

The theoretical framework that I set up in order to interpret the experiments allows for an interesting additional interpretation of the subjects's behavior, already known to be adaptive in general terms, as shown in [15]. For one thing, randomness itself would be generated by the fact that the experimental agents would care more about what they have personally witnessed in the past than about what they are told (and they know is true), even though this information might be more revealing about the future. This interpretation of the data points to a further level of bounded rationality in experimental subjects more generally.

In particular, I have set up a framework for the OLG model without productivity or preference shocks, in order to be able to consider and compare to experimental data the aggregate effects of individual differences between the participating agents. The framework includes beliefs, information sets and a model of time participation. *AA*-kind of models are justified in this framework as models that describe the experimental features of learning and equilibrium selection. It is shown that if agents use simple (as opposed to mixed or experimenting) adaptive rules, representative agent *AA* models *à la* Marcet–Sargent are not appropriate to describe the uncertainty inherent in economic experiments in which different individuals participate. It is shown, furthermore, that heterogeneity in learning rules is not sufficient in these kinds of models to deliver stochastic-like time paths if agents participate at every period in the economy. More importantly, it is shown that if agents have different turns to participate and use simple adaptive rules, the only way for the economy to show erratic oscillations over time is for agents not to use only public information to form their estimations when they use their learning algorithms. The use of data *experienced* when agents participate, even though reflecting further lack of rationality, is shown to be essential to

¹⁹I also suggest a way of including in the model random guesses.

produce the inherent uncertainty present in the data, under the assumption that agents themselves do not experiment when making decisions.

In a positive light, these facts show that there is nothing inherently bad about the *AA*-kind of models to reproduce experimental data and pinpoint, at the same time, correct modifications in order to explain reality. Although the *GA* has been put forward in [3] to model, in an entirely different way, the experimental features of erratic oscillations and equilibrium selection, we highlight that, as it is, that model requires a kind of individual interaction not present in the experimental environments. Our version of the *AAs* adequately reproduces those experimental features, while keeping the assumptions on agent behavior close to the experimental design. It is that reproduction feature what gives strength to the further bounded rationality interpretation, shedding light for modeling behavior on different settings. The participation model itself reproduces an important feature of real life, and might be of use in other contexts. To be sure, it might be useful in testing for the utilization of private *vs* public information by subjects, as far as we are able to control for experimentation.

Simulations of our examples of adaptive models are presented in order to show how the model reproduces the experimental features of equilibrium selection and erratic oscillations. In addition, the examples provided illustrate the kind of calibration exercise one could perform using the model, by modifying optimally learning rates, initial conditions and participation times. In the first example, agents are assumed to estimate the next period's price using *OLS* with past prices and making savings decisions in order to maximize the utility that comes from believing the projected price. In simulations we compare the case in which agents use all publicly available prices in their forecast, (the representative agent model, it turns out), with the one in which they use only the *experienced* ones. As expected from the theory, the time paths in the first case are smooth, while in the second are erratic, but converging to the right limit. In the second and third examples, people do not maximize but adjust savings as if approximating maximization, while having myopic beliefs about inflation rates. These two last examples differ only in the complexity of the calculations needed to make a decision. The third example is particularly simple in this regard, and reproduces basically the same kind of time paths in simulations as the second, in a wide variety of circumstances.

We use simulations in order to illustrate that the last two examples have more richness than the *OLS* model. The richness refers to the fact that their underlying deterministic process is able to show, alternatively, monotonic or persistent oscillatory behavior, depending on the learning rates chosen. This contrasts with the *OLS* model, which shows only monotonic convergent paths after a couple of initial oscillations.

Our conclusions call for experiments in order to perform a more careful test of the use of private experience versus public information. A possible way to do this is to allow subjects only to use some proposed rules (one fixed per individual), considered realistic from other studies, and set up the random participation, as in the Marimon-Sunder experiments. Incidentally, even though the analysis was done in an overlapping generations model, the analytical conclusions easily extend to models with infinitely lived agents and cash-in-advance constraints. An experimental design for this case, though, remains to be carried out.

A Appendix: About the Simulations

In the simulations I will use the parameters of the experiment number 2 performed in [15]. As we know, they perform a total of 13 experiments, and in this case, the number of participants is 12 people ($N = 12$), out of which 3 are chosen, at random, to participate as young every period ($n = 3$), making sure nobody lives two lives in a row. The endowments of chips are 7 when young, and 1 when old ($W_1 = 7$, $W_2 = 1$); the amount of money given to each initial old is 3.722 ($h_0 = 3.722$); the real deficit per young is 1.25 every period ($d = 1.25$) and the discount rate is 1 ($\beta = 1$). With these parameters, the *LSS* savings per capita and inflation rate under the *REE* assumptions are, respectively, $s^L = 2.50$ and $\pi^L = 2$. The corresponding values for the *HSS* equilibrium are, respectively, $s^H = 1.75$ and $\pi^H = 3.50$. The values for these inflation rates can be seen intercepting the 45 degree line in Figure 1, where the typical Rational Expectations first order difference equation in those variables is represented, for the above parameters.

The simulations were generated by a Matlab program that follows the indications outlined in the formalizations presented in previous sections. In particular, versions of the *OLS*, *NAA* and *DSE* examples are simulated. Participation times are dictated by an initial ω , which is a $2n$ by T (the total number of periods) matrix indicating in the i, j entry the individual in the set $\{1, 2, \dots, N\}$ who participates in period j : if $i \in \{1, 2, \dots, n\}$ as an old, and if $i \in \{n + 1, n + 2, \dots, 2n\}$ as a young person. Young people are chosen every period, each individual with equal probability of being chosen, among the $N - 2n$ people that are not old in that period, and were not old in the previous one, in order to follow the experimental design. Once a program that determines the stopping times and cumulative turn indexes is run, the economy develops, after random initial savings decisions: At any given period, each selected agent uses the algorithm of the example being simulated, when his or her time to participate arrives, incorporating the private or public experience accumulated in their decision making, according to the assumption in place. Markets then clear (government's current monetary deficit plus old people's money supply equals young people's money demand for savings), producing an equilibrium price. Next period, the newly selected agents make their decisions, and the process repeats itself again and again, until the last period, which nobody knows before hand.

After many simulations, we observe that our versions are less *stable* than versions with homogeneous agents, in the sense that there are initial conditions that guarantee convergence in the models with homogeneous agents, but that generate divergent sequences in the models with heterogeneous agents. Even though diverging cases where very rare, this feature is a cost we have to pay in order to be able to reproduce a much more complex and unstable world than the one explained by the previous models that show convergence, but smooth time paths, or even the ones that use experimentation with controlled errors.

If one wanted to do calibration to the experimental data, learning rates, initial conditions and participation times could be chosen optimally to fit the data, for a given method, and the goodness of fit, if consistent through all the experiments, could be used to argue for one method over the others as a model of actual behavior. This is illustrated in [17]. Unfortunately, individual participation dates and individual initial conditions, which are fundamental for calibration in models of heterogeneous agents like ours, were not available to us, in order to be able to perform the exercise.

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